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Managing Pension Assets: from Surplus Optimization to Liability-Driven Investment

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Abstract

In this paper, we consider an intertemporal portfolio problem in the presence of liability constraints. Using the value of the liability portfolio as a natural numeraire, we find that the solution to this problem involves a three fund separation theorem that provides formal justification to some recent so-called liability-driven investment solutions offered by several investment banks and asset management firms, which are based on investment in two underlying building blocks (in addition to the risk-free asset), the standard optimal growth portfolio and a liability hedging portfolio.

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1. Introduction

Recent difficulties have drawn attention to the risk management practices of institutional investors in general and defined benefit pension plans in particular. What has been labeled “a perfect storm” of adverse market conditions over the past three years has devastated many corporate defined benefit pension plans. Negative equity market returns have eroded plan assets at the same time as declining interest rates have increased market-to-market value of benefit obligations and contributions. In extreme cases, this has left corporate pension plans with funding gaps as large as or larger than the market capitalization of the plan sponsor. For example, in 2003, the companies included in the S&P 500 and the FTSE 100 index faced a cumulative deficit of $225 billion and £55 billion, respectively (Credit Suisse First Boston (2003) and Standard Life Investments (2003)), while the worldwide deficit reached an estimated 1,500 to 2,000 billion USD (Watson Wyatt (2003)).

That institutional investors in general and pension funds in particular have been so dramatically affected by recent market downturns has emphasized the weakness of risk management practices. In particular, it has been argued that the kinds of asset allocation strategies implemented in practice, which used to be heavily skewed towards equities in the absence of any protection with respect to their downside risk, were not consistent with a sound liability risk management process.

In this context, new approaches that are referred to as *liability driven investment* ("LDI") solutions have rapidly gained interest with pension funds, insurance companies, and investment consultants alike, following recent changes in accounting standards and regulations that have led to an increase focus on liability risk management. While their promoters argue that such LDI strategies can add significant value in terms of liability risk management, their benefits from a rational standpoint have not been documented in the academic literature and a significant number of institutional investors are still reluctant to use them.

The aim of this paper is to provide an academic perspective on asset-liability management (ALM) strategies. In particular, we introduce a formal continuous-time model of intertemporal asset allocation decisions in the presence of liability constraints, and discuss how recent industry trends such as liability-driven investment fit with respect to the theoretical optimally designed strategies. The rest of the paper is organized as follows. In section 2, we provide a brief history of ALM techniques, outlining both the practitioner and the academic standpoints. In section 3, we introduce a formal model of asset-liability management. In section 4, we present a conclusion.

2. A (Very) Brief History of ALM

Asset-Liability Management (ALM) denotes the adaptation of the portfolio management process in order to handle the presence of various constraints relating to the commitments that figure in the liabilities of an institutional investor's balance sheet (commitments to paying pensions, insurance premiums, etc.). There are therefore as many types of liability constraints as there are types of institutional investors, and thus as many types of approaches to asset-liability management.

In what follows, we will provide a brief review of standard techniques used in ALM, both from a practitioner's perspective and from an academic perspective.

2.1. ALM from a Practitioner's Perspective

From a practical standpoint, ALM-type management techniques can be classified into several categories.

A first approach called *cash-flow matching* involves ensuring a perfect *static* match between the cash flows from the portfolio of assets and the commitments in the liabilities. Let us assume for example that a pension fund has a commitment to pay out a monthly pension to a retired person. Leaving aside the complexity relating to the uncertain life expectancy of the retiree, the structure of the liabilities is defined simply as a series of cash outflows to be paid, the real value of which is known today, but for which the nominal value is typically matched with an inflation index. It is possible in theory to construct a portfolio of assets whose future cash flows will be identical to this structure of commitments. To do so, assuming that securities of that kind exist on the market, would involve purchasing inflation-linked zero-coupon bonds with a maturity corresponding to the dates on which the monthly pension installments are paid out, with amounts that are proportional to the amount of real commitments.

This technique, which provides the advantage of simplicity and allows, in theory, for perfect risk management, nevertheless presents a number of limitations. First of all, it will generally be impossible to find inflation-linked securities whose maturity corresponds exactly to the liability commitments. Moreover, most of those securities pay out coupons, which leads to the problem of reinvesting the coupons. To the extent that perfect matching is not possible, there is a technique called *immunization*, which allows the residual interest rate risk created by the imperfect match between the assets and liabilities to be managed in a *dynamic* way. This interest rate risk management technique can be extended beyond a simple duration-based approach to fairly general contexts, including for example hedging larger changes in interest rates (through the introduction of a convexity adjustment), hedging non-parallel shifts in the yield curve (see for example Fabozzi, Martellini and Priaulet (2005)), or to simultaneous management of interest rate risk and inflation risk (Siegel and Waring (2004)). It should be noted, however, that this technique is difficult to adapt to hedging non-linear risks related to the presence of options hidden in the liability structures, and/or to hedging non-interest rate related risks in liability structures.

Another, probably more important, disadvantage of the cash-flow matching technique (or of the approximate matching version represented by the immunization approach) is that it represents a positioning that is extreme and not necessarily optimal for the investor in the risk/return space. In fact we can say that the cash-flow matching approach in asset-liability management is the equivalent of investing in the risk-free asset in an asset management context. It allows for perfect management of the risks, namely a capital guarantee in the passive management framework, and a guarantee that the liability constraints are respected in the ALM framework. However, the lack of return, related to the absence of risk premia, makes this approach very costly, which leads to an unattractive level of contribution to the assets.

In a concern to improve the profitability of the assets, and therefore to reduce the level of contributions, it is necessary to introduce asset classes (stocks, government bonds and corporate bonds) which are not perfectly correlated with the liabilities into the strategic allocation. It will then involve finding the best possible compromise between the risk (relative to the liability constraints) thereby taken on, and the excess return that the investor can hope to obtain through the exposure to rewarded risk factors. Different techniques are then used to optimize the surplus, i.e., the excess value of the assets compared to the liabilities, in a risk/return space. In particular, it is useful to turn to stochastic models that allow for a representation of the uncertainty relating to a set of risk factors that impact the liabilities. These can be financial risks (inflation, interest rate, stocks) or non-financial risks (demographic ones in particular). When necessary, agent behavior models are then developed, which allows the impact of decisions linked to the exercising of certain implicit options to be represented. For example, an insured person can (typically in exchange for penalties) cancel his/her life assurance contract if the guaranteed contractual rate drops significantly below the interest rate level prevailing at a date following the signature of the contract, which makes the amount of liability cash flows, and not just their current value, dependent on interest rate risk.

It is also appropriate to mention non-linear risk-profiling management techniques, the goal of which is to provide a compromise between a risk-free and return-free approach on the one hand, and a risky approach that does not allow the liability constraints to be guaranteed on the other (see the table below for an overview of ALM techniques and the corresponding techniques in asset management). In particular, it involves introducing options which allow for (partial) access to the risk premia of stocks without all of the associated risks, or dynamic allocation methods, inspired by the portfolio insurance techniques transposed into an ALM framework (see in particular Leibowitz and Weinberger (1982ab) for the *contingent optimisation* technique, or Amenc, Malaise and Martellini (2004) for a generalisation in terms of a *dynamic core-satellite* approach).

Asset Management (absolute risk)	***Risk/Return profile***	***Asset-Liability Management (relative risk)***
Investment in the risk-free asset	Zero risk - no risk premia	Cash-flow matching and/or immunisation
Diversified portfolio including risky assets	Optimal risk/return compromise	Optimisation of the surplus
Portfolio insurance	Dynamic and skewed management	Contingent immunisation

Finally, it is appropriate to mention a new approach that is referred to as *liability driven investment* ("LDI"). This is an approach that has rapidly gained interest with pension funds, insurance companies, and investment consultants alike, following recent changes in accounting standards and regulations that have led to an increase focus on liability risk management. Essentially, these changes force institutional investors to value their liabilities at market rates (mark-to-market), instead of fixed discount rates, which results in an increase of the liability portfolio volatility. As a result, institutional investors have to increase their focus on risk management to reduce the volatility of their funding ratio, a new constraint reinforced by stricter solvency requirements. While they can vary significantly across providers, LDI solutions typically involve a hedge of the duration and convexity risks via seven standard building blocks, while keeping some assets free for investing in higher yielding asset classes.

These solutions may or may not involve leverage, depending on the institutional investor's risk aversion. When no leverage is used, a fraction of the assets (known as the *liability-matching portfolio*) is allocated to risk management, while another fraction of the asset is allocated to performance generation. One may actually view this approach as a combination of two strategies, involving investing in immunization strategies (for risk management) as well as investing in standard asset management solutions (for performance generation). This approach stands in sharp contrast to more traditional surplus optimization methods, where both objectives (liability risk management and performance generation) are pursued simultaneously in an attempt to achieve the portfolio with the highest possible relative risk/relative return ratio. When leverage is used, it can be explicit, under the form of a short position in the risk-free asset, or implicit, under the form of leverage induced by the use of derivatives (typically interest rates and/or inflation swaps) in the liability-matching portfolio. This allows for more potential for performance generation. For example, one may consider a stylized example where derivatives are used to match the liability portfolio so that virtually 100% of the assets are still available for investment in the performance generation portfolio. It should be noted that the performance target for this "risky" portfolio then becomes the risk-free rate, which legitimates the used of absolute return portfolios (hedge funds, capital guaranteed products, etc.).

In section 3, we will actually argue that this allocation approach, expressed in terms of allocation to three building blocks (cash, liability-matching portfolio, and performance portfolio), as opposed to allocation to standard asset classes, is consistent with a *three-funds separation theorem* that extends standard results from modern portfolio theory to situations involving the presence of liability constraints, and constitutes a first step forward a better asset-liability management process.

2.2. ALM from an Academic Perspective

While it seems that a variety of techniques are a priori available to institutions who seek to manage their asset portfolio in the face of their liability constraints, it remains to be seen what results, if any, are available from an academic perspective about the optimality, or lack thereof, of these various approaches to ALM. The existing contributions in the academic literature fall within two different, and somewhat competing, approaches, to ALM.

On the one hand, several authors have attempted to cast the ALM problem in a continuous-time framework, and extend Merton's intertemporal selection analysis (see Merton (1969, 1971)) to account for the presence of liability constraints in the asset allocation policy. A first step in the application of optimal portfolio selection theory to the problem of pension funds has been taken by Merton (1990) himself, who studies the allocation decision of a University that manages an endowment fund. In a similar spirit, Boulier et al. (1995) have formulated a continuous-time dynamic programming model of pension fund management. It contains all of the basic elements for modeling dynamic pension fund behavior, and can be solved by means of analytical methods.[1] Rudolf and Ziemba (1994) extend these results to the case of a time-varying opportunity set, where state variables are interpreted as currency rates that affect the value of the pension's asset portfolio. Also related is a paper by Sundaresan and Zapatero (1997), which is specifically aimed at asset allocation and retirement decisions in the case of a pension fund. This continuous-time stochastic control approach to ALM is appealing because it enjoys the desirable property of tractability and simplicity, allowing one to fully and explicitly understand the various mechanisms affecting the optimal allocation strategy.

On the other hand, because of the simplicity of the modelling approach, such continuous-time models do not allow for a full and realistic account of uncertainty facing institutions in the context of asset-liability management. A second strand of the literature has therefore focused on developing more comprehensive models of uncertainty in an ALM context. This has led to the development of a stochastic programming approach to ALM, including Kallberg *et al*. (1982), Kusy and Ziemba (1986), or Mulvey and Vladimirou (1992). This strand of the literature is relatively close to industry practice, with one of the first successful commercial multistage stochastic programming applications appearing in the Russell-Yasuda Kasai Model (Cariño *et al*. (1994, 1998), Cariño and Ziemba (1998). Other successful commercial applications include the Towers Perrin-Tillinghast ALM system of Mulvey *et al*. (2000), the fixed-income portfolio management models of Zenios (1995) and Beltratti *et al*. (1999), and the InnoALM system of Geyer *et al*. (2001). A good number of applications in asset-liability management are provided in Ziemba and Mulvey (1998) and Ziemba (2003). In most cases, stochastic programming models require the uncertainties be approximated by a scenario tree with a finite number of states of the world at each time. Important practical issues such as transaction costs, multiple state variables, market incompleteness due to uncertainty in liability streams that is not spanned by existing securities, taxes and trading limits, regulatory restrictions and corporate policy requirements can be handled within the stochastic programming framework. On the other hand, this comes at the cost of tractability. Analytical solutions are not possible, and stochastic programming models need to be solved via numerical optimization. In an attempt to circumvent the concern of the back-box flavor of stochastic programming models, some interesting attempts have been made to test for the optimality of various rule-based strategies (see Mulvey et al. (2005)).

[1] A related reference is Siegmann and Lucas (2002) who extend the approach taken by Boulier et al. (1995) by considering CARA and CRRA preferences, as opposed to a simple quadratic loss function.

In the next section, we introduce a stylized continuous-model for intertemporal allocation decisions in the presence of liability constraints, which falls within the first strand of the literature. Under specific assumptions, we will be able to provide explicit solutions, and show that a three fund separation theorem holds that can be related to the recent LDI approach to ALM.

3. A Formal Continuous-Time Model of Asset-Liability Management

In this section, we introduce a general model for the economy in the presence of liability constraints. Let *[0,T]* denote the (finite) time span of the economy, where uncertainty is described through a standard probability space *(Ω,A,P)* and endowed with a filtration $\{F_t; t \geq 0\}$, where $F_\infty \subset A$ and F_0 is trivial, representing the *P*-augmentation of the filtration generated by the *n*-dimensional Brownian motion $\left(W^1,\ldots,W^n\right)$.

3.1. Stochastic Model for the Value of Asset and Liabilities

We consider *n* risky assets (or asset classes), the prices of which are given by :

$$dP_t^i = P_t^i\left(\mu_i dt + \sum_{j=1}^{n}\sigma_{ij} dW_t^j\right), i = 1,\ldots,n$$

We shall sometimes use the shorthand vector notation for the expected return (column) vector $\mu = \left(\mu_i\right)'_{i=1,\ldots,n}$ and matrix notation $\sigma = \left(\sigma_{ij}\right)_{i,j=1,\ldots,n}$ for the asset return variance-covariance matrix. We also denote **1**=(1,…,1)' a n-dimensional vector of ones and by $W = \left(W^j\right)'_{j=1,\ldots,n}$ and the vector of Brownian motions. A risk-free asset, the 0^{th} asset, is also traded in the economy. The return on that asset, typically a default free bond, is given by $dP_t^0 = P_t^0 r dt$, where *r* is the risk-free rate in the economy.

We assume that r, μ and σ are progressively-measurable and uniformly bounded processes, and that σ is a non singular matrix that is also progressively-measurable and bounded uniformly.[2] For some numerical applications below, we will sometimes treat these parameter values as constant.

We also introduce a separate process that represents in a reduced-form manner the dynamics of the present value of the liabilities:

$$dL_t = L_t\left(\mu_L dt + \sum_{j=1}^{n} \sigma_{L,j} dW_t^j + \sigma_{L,\varepsilon} dW_t^\varepsilon\right)$$

where $\left(W_t^\varepsilon\right)$ is a standard Brownian motion, uncorrelated with W, that can be regarded as the projection residual of liability risk onto asset price risk and represent the source of uncertainty that is specific to liability risk, emanating from various factors such as uncertainty in the growth of work force, uncertainty in mortality and retirement rates, etc.

The integration of the above stochastic differential equation gives: $L_T = L_t \eta(t,T) \eta_L(t,T)$, with:

$$\eta(t,T) \equiv \exp\left\{\int_t^T \left(\mu_L(s) - \frac{1}{2}\sigma_L^{'}(s)\sigma_L(s)\right)ds + \int_t^T \sigma_L^{'}(s) dW_s\right\}$$
$$\eta_L(t,T) \equiv \exp\left\{-\int_t^T \frac{1}{2}\sigma_{L,\varepsilon}^2(s)ds + \int_t^T \sigma_{L,\varepsilon}(s) dW_s^\varepsilon\right\}$$

When $\sigma_{L,\varepsilon} = 0$, then we are in a complete market situation where all liability uncertainty is spanned by existing securities. Because of the presence of non-financial risks (e.g., actuarial risks), such a situation never occurs in practice, and the correlation between the liability and the liability-hedging portfolio (i.e., the portfolio with the highest correlation with liability values) is always strictly lower than one. In general therefore, $\sigma_{L,\varepsilon} = 0$ and the presence of liability risk that is not spanned by asset prices induces a specific form of market incompleteness.

[2] More generally, one can make expected return and volatilities of the risky assets, as well as the risk-free rate, depend upon a multi-dimensional state variable X. These states variables can be thought of various sources of uncertainty impacting the value of assets and liabilities. In particular, one may consider the impact of stochastic interest rate on the optimal policy.

3.2. Objective and Investment Policy

We now introduce a couple of variables of interest, which will be used as a state variable in this model, is the surplus. The first one is the surplus, defined as the difference in value between assets and liabilities: $S_t = A_t - L_t$; the second one is the funding ratio, defined as the ratio of assets to liabilities: $F_t = A_t / L_t$. A pension trust has a surplus when the surplus is greater than zero (funding ratio > 100%), fully funded when it is zero (funding ratio = 100%), and under funded when it is less than zero (funding ratio < 100%). In an asset-liability management context, what matters is not the value of the assets per se, but how the asset value compares to the value of liabilities. This is also the reason why it is natural to assume that the (institutional) investor's objective is written in terms of relative wealth (relative to liabilities), as opposed to absolute wealth: $\max_w E_0\left[U(F_T)\right]$.

The investment policy is a (column) predictable process vector $\left(w_t^{'} = (w_{1t},\ldots,w_{nt})\right)_{t\geq 0}$ that represents allocations to risky assets, with the reminder invested in the risk-free asset. We define by A_t^w the asset process, i.e., the wealth at time *t* of an investor following the strategy *w* starting with an initial wealth A_0.

We have that:

$$dA_t^w = A_t^w\left[(1 - w'.\mathbf{1})\frac{dB_t}{B_t} + w'\frac{dP_t}{P_t}\right]$$

or:

$$dA_t^w = A_t^w\left[(r + w'(\mu - r\mathbf{1}))dt + w'\sigma dW_t\right]$$

Using Itô's lemma, we can also derive the stochastic process followed by the funding ratio under the assumption of a strategy *w*:

$$dF_t^w = d\left(\frac{A_t^w}{L_t}\right) = \frac{1}{L_t}dA_t^w - \frac{A_t^w}{L_t^2}dL_t - \frac{1}{L_t^2}dA_t^w dL_t + \frac{A_t^w}{L_t^3}(dL_t)^2$$

which yields:

$$\frac{dF_t^w}{F_t^w} = \left(\left(r + w'\left(\mu - r\mathbf{1}\right)\right)dt + w'\sigma dW_t\right) - \left(\mu_L dt + \sigma_L^{'} dW_t + \sigma_{L,\varepsilon} dW_t^{\varepsilon}\right) - \left(w'\sigma\sigma_L dt\right) + \left(\left(\sigma_L^{'}\sigma_L dt + \sigma_{L,\varepsilon}^2\right)dt\right)$$

or

$$\frac{dF_t^w}{F_t^w} = \left(r - \mu_L + \sigma_L^{'}\sigma_L + \sigma_{L,\varepsilon}^2\right)dt + w'\left(\left(\mu - r\mathbf{1}\right) - \sigma\sigma_L\right)dt + \left(w'\sigma - \sigma_L^{'}\right)dW_t - \sigma_{L,\varepsilon} dW_t^{\varepsilon}$$

For later use, let us define the following quantities as the mean return and volatility of the funding ratio portfolio, subject to a portfolio strategy *w*:

$$\begin{aligned}
\mu_F^w &\equiv \left(r - \mu_L + \sigma_L^{'}\sigma_L + \sigma_{L,\varepsilon}^2\right) + w'\left(\left(\mu - r1\right) - \sigma\sigma_L\right) \\
\sigma_F^w &\equiv \left(\left(w'\sigma - \sigma_L^{'}\right)^{'}\left(w'\sigma - \sigma_L^{'}\right) + \sigma_{L,\varepsilon}^2\right)^{1/2}
\end{aligned}$$

3.3. Solution using the Dynamic Programming Approach

Define the indirect or derived utility process at time *t*:

$$J_t = \max_w E_t\left[U\left(F_T\right)\right]$$

where $E_t[\bullet]$ denotes the expectation conditional about information available at time *t*, such as described by the filtration generated by the *n* Brownian motion driven asset prices and the *(n+1)*th Brownian motion driving pure liability uncertainty.

3.3.1. General Solution

For a Markovian control process $\left(w_t\right)_{t\geq 0}$ and a function $\varphi\left(t, F_t\right) \in C^{1,2}$ the infinitesimal generator of the funding ratio process is:

$$A^w\varphi(t,F_t)=\varphi_t+F\varphi_F\mu_F^w+\tfrac{1}{2}F^2\varphi_{FF}\left(\sigma_F^w\right)^2$$

where the derivative of a function *f* with respect to variable *x* is denoted as f_x.

Given the objective function in, the appropriate Hamilton-Jacobi-Bellman equation associated with this problem is:

$$\sup_w\left\{A^wJ(t,F_t)\right\}=0$$

subject to $J(T,F_T)=U(F_t)$.

Optimizing with respect to *w* yields:

$$F\varphi_F\frac{\partial\mu_F^w}{\partial w}\left(w^*\right)+\tfrac{1}{2}F^2\varphi_{FF}\frac{\partial\left(\sigma_F^w\right)^2}{\partial w}\left(w^*\right)=0$$

or:

$$F\varphi_F\left((\mu-r\mathbf{1})-\sigma\sigma_L\right)+F^2\varphi_{FF}\left(\left(w^{*'}\sigma\sigma'-\sigma\sigma_L\right)\right)=0$$

with solution:

$$w^*=w^*(t,F_t)=-(\sigma\sigma')^{-1}\left((\mu-r\mathbf{1})-\sigma\sigma_L\right)\frac{\varphi_F(t,F_t)}{F\varphi_{FF}(t,F_t)}+(\sigma\sigma')^{-1}\sigma\sigma_L$$

or:

$$w^*=w^*(t,F_t)=-(\sigma\sigma')^{-1}(\mu-r\mathbf{1})\frac{\varphi_F(t,F_t)}{F\varphi_{FF}(t,F_t)}+\left(1+\frac{\varphi_F(t,F_t)}{F\varphi_{FF}(t,F_t)}\right)(\sigma')^{-1}\sigma_L$$

We thus obtain a three funds separation theorem, where the optimal portfolio strategy consists of holding two funds, one with weights $w_M = \frac{(\sigma\sigma')^{-1}(\mu - r\mathbf{1})}{\mathbf{1}'(\sigma\sigma')^{-1}(\mu - r\mathbf{1})}$ and another one with weights $w_L = \frac{(\sigma')^{-1}\sigma_L}{\mathbf{1}'(\sigma')^{-1}\sigma_L}$, the rest being invested in the risk-free asset.

The first portfolio is the standard mean-variance efficient portfolio. Note that the amount invested in that portfolio is directly proportional to the investor's Arrow-Pratt coefficient of risk-tolerance $-\frac{\varphi_F}{F\varphi_{FF}}$ (the inverse of the relative risk aversion). This makes sense: the higher the investor's (funding) risk tolerance, the higher the allocation to that portfolio will be.

In order to better understand the nature of the second portfolio, it is useful to remark that it is a portfolio that minimizing the local volatility σ_F^w of the funding ratio. To see this, recall that the expression for the local variance is given by $\sigma_F^w = \left((w'\sigma - \sigma_{L^2}^{'})^{'}(w'\sigma - \sigma_L^{'}) + \sigma_{L,\varepsilon}^2\right)^{1/2}$, which reaches a minimum for $w^* = (\sigma')^{-1}\sigma_L$, with the minimum being $\sigma_{L,\varepsilon}^2$. As such, it appears as the equivalent of the minimum variance portfolio in a relative return-relative risk space, also the equivalent of the risk-free asset in a complete market situation where liability risk is entirely spanned by existing securities ($\sigma_{L,\varepsilon}^2 = 0$). Alternatively, this portfolio can be shown to have the highest correlation with the liabilities. As such, it can be called a liability-hedging portfolio, in the spirit of Merton (1971) intertemporal hedging demands. Indeed, if we want to maximize the covariance $w'\sigma\sigma_L$ between the asset portfolio and the liability portfolio *L*, under the constraint that $\sigma_A^2 = w'\sigma\sigma' w$, we obtain the following Lagrangian: $L = w'\sigma\sigma_L - \lambda(w'\sigma\sigma' w - \sigma_A^2)$.

Differentiating with respect to *w* yields: $\frac{\partial L}{\partial w} = \sigma\sigma_L - 2\lambda\sigma'\sigma w$, with a strictly negative second derivative function. Setting the first derivative equal to zero for the highest covariance portfolio leads to the following portfolio, which is indeed proportional to the liability hedging portfolio $w = \frac{1}{2\lambda}(\sigma\sigma')^{-1}\sigma\sigma_L = \frac{1}{2\lambda}(\sigma')^{-1}\sigma_L$.

3.3.2. Specific Solution in Case of CRRA Utility and Constant Parameter Values

Let us now consider a specific utility function of the CRRA type:

$$U(F_T) = \frac{(F_T)^{1-\gamma}}{1-\gamma}$$

We try a solution to the non linear Cauchy problem

$$\varphi_t + F\varphi_F \mu_F^{w^*} + \tfrac{1}{2} F^2 \varphi_{FF} \left(\sigma_F^{w^*}\right)^2 = 0$$

which is separable in F and can be written as:

$$\varphi(t, F_t) = \frac{g(t,T)(F_t)^{1-\gamma}}{1-\gamma}$$

with

$$g(t,T) = \exp\left[(T-t)\left(-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)(\theta-\sigma_L)'(\theta-\sigma_L) - \gamma\left(\sigma_{L,\varepsilon}^2 - \theta_L \sigma_{L,\varepsilon}\right) + \frac{(1-\gamma)(2-\gamma)}{2}\sigma_{L,\varepsilon}^2\right)\right]$$

where $\sigma - \sigma_L$ is defined as the matrix which general term is equal to that of σ outside the diagonal and is equal to $\sigma_{ii} - \sigma_L$, also written as $\sigma_i^2 - \sigma_L$, on the diagonal.

Given that $-\dfrac{J_F}{FJ_{FF}} = \dfrac{1}{\gamma}$, we finally obtain:

$$w^* = w^*(t, F_t) = \frac{1}{\gamma}(\sigma\sigma')^{-1}(\mu - r\mathbf{1}) + \left(1-\frac{1}{\gamma}\right)(\sigma')^{-1}\sigma_L$$

As is well-known, it should be noted that when $\gamma = 1$, i.e., in the case of the log investor, the intertemporal hedging demand is zero (myopic investor).

In general, again, the optimal strategy consists of holding two funds, in addition to the risk-free asset, the standard mean-variance portfolio and the liability hedging portfolio, and the proportions invested in these two funds are constant in time.

Note also that, as outlined in the previous sections, several investment banks have suggested using customized derivatives to perform liability-matching, and use leverage so that full amount of asset portfolio is still invested in a risky asset. This strategy corresponds to -100% in cash, 100% in liability-hedging portfolio and 100% in market portfolio, which can be rationalized under a specific choice of the risk aversion coefficient. More risk-averse investors, on the other hand, will prefer solutions involving less or no leverage.

4. Conclusion

In this paper, we have considered an intertemporal portfolio problem in the presence of liability constraints. Using the value of the liability portfolio as a natural numeraire, we have found that the solution to this problem involves a three fund separation theorem that provides formal justification to some recent so-called liability-driven investment solutions offered by several investment banks and asset management firms, which are based on investment in two underlying building blocks (in addition to the risk-free asset), the standard optimal growth portfolio and a liability hedging portfolio.

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